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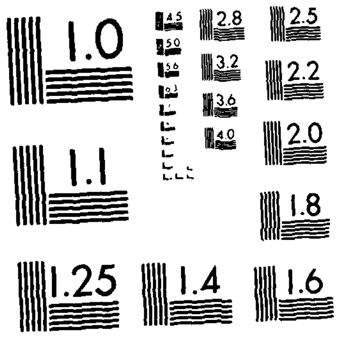

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MOMENTS OF THE MINIMUM OF A RANDOM WALK
AND COMPLETE CONVERGENCE

BY

MICHAEL HOGAN

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Abbreviated Title: Moments of Minimum of a Random Walk

Key Words: Random Walk, Renewal Theorem, Complete Convergence, Strong Law of Large Numbers

Summary: Moments of the Minimum of a Random walk and Complete Convergence. Let S_n be a random walk with positive drift. Let $S_{min} = \inf_{n>0} \{S_n\}$. New proofs are given of the following: For $p \geq 1$ $E|S_{min}|^p < \infty \iff E(|S_1|^{p+1} 1_{(S_1<0)}) < \infty$; $\Sigma P\{S_n < 0\} < \infty \iff E(|S_1|^2 1_{(S_1<0)}) < \infty$, and some related results.

1. Introduction

This paper gives new proofs of the equivalences that are stated as Theorem 1 in Section 3. Robbins and Hsu [6] first showed c => e in 1948 with $p = 1$. They considered the problem in the context of a random walk generated by X_i with $E X_i = 0$, and showed that $E X_i^2 < \infty$ $\Rightarrow \sum_n P\{\frac{|S_n|}{n} > \varepsilon\} < \infty$. They called the finiteness of this sum complete convergence. It implies the strong law of large numbers by an application of the Borel-Cantelli lemma. Erdos [3] proved the reverse direction in 1949 and Baum and Katz [1] added the equivalence of (d) in 1965. Kiefer and Wolfowitz [5] estiblished the equivalence of (c) and (f) and the (c)<=>(g) is in Taylor [7]. Independent discovery of both of the results were credited by the respective authors to unpublished work of Darling, Erdos and Kakutani. These results are partially restated as Theorems 2 and 3 of section 4.

The new proofs provide an ε-free approach to these problems. The elementary Renewal Theorem, time reversal, and Wald's identities are the primary tools, and suffice for the case $p=2$. For larger p, the martingale conditional square function has to be used to replace Wald's identities to show the existence of moments in stopped random walks.

2. Notation and Conventions.

Fix the following notation and conventions. X_i is an i.i.d. sequence with $\mu = EX_i > 0$; $X^- = -X\, 1_{\{X<0\}}$; $S_0 = 0$, $S_i = \sum_{j=1}^{i} X_j$ for $i > 0$; $S_{min} = \inf\{S_i;\ i > 0\}$; τ_+ is the first strict ascending ladder epoch, $\tau_+^{(j)}$ is the j^{th} strict ascending ladder epoch: τ_- is the first weak descending ladder epoch, or $+\infty$ if none exists, $\tau_-^{(j)}$ is the jth weak descending ladder epoch, or $+\infty$ if none exists (see Feller [4], Sec. 12.1 for definition);

$$L(0) = \sum_{n=1}^{\infty} 1_{\{\inf_{j\geq n} S_j \leq 0\}}, \quad N(0) = \sum_{n=1}^{\infty} 1_{\{S_n \leq 0\}},$$

i.e. $L(0)$ is the last time the process is non-positive, and $N(0)$ is the number of times the process is non-positive; $\inf\{\ \} = \infty$; $\tau(a) = \inf\{n > 0\colon\ S_n \geq a\}$; $t(a) = \inf\{n > 0\colon\ S_n \leq a\}$; K and C will be positive constants, not necessarily the same from line to line; $E\{Y;A\} = E(Y1_A)$; $\omega = (X_1, X_2, \ldots)$, $\omega_a^+ = (X_{a+1}, X_{a+2}, \ldots)$. E^x denotes expectation of the random walk started from x; $E = E^0$.

3. Statement and Proof of Theorem:

Theorem 1: For $p \geq 2$ the following are equivalent:

a). $E(\tau_+{}^p) < \infty$

b). $E(\tau_-{}^{p-1};\ \tau_- < \infty) < \infty$

c). $E((X^-)^p) < \infty$

d). $E(L(0)^{p-1}) < \infty$

e). $E(N(0)^{p-1}) < \infty$

f). $E(|S_{min}|^{p-1}) < \infty$

g). $E(|S_{\tau_-}|^{p-1};\ \tau_- < \infty) < \infty$.

Four lemmas will be given first, then the proof proceeds as follows: a <=> b; a <=> c; d => b; a and b => d; d => e; e => g; g => f; f => e; f => c.

Lemma 1: If $E(\tau_+{}^p) < \infty$ then $E(t(-x)^{p-1};\ t(-x) < \infty) < K \quad \forall\ x > 0$.

Proof: By time-reversal one has, $\forall\, K > 0$,

$$P\{t(-K) = n\} = P\{S_1 > -K,\dots,S_{n-1} > -K,\ S_n < -K\}$$

$$= P\{S_n - S_{n-1} > -K,\dots,S_n - S_1 > -K,\ S_n < -K\}$$

$$\le P\{S_1 < 0,\dots,S_n < 0\}$$

$$= P\{\tau_+ > n\}.$$

Multiply the first and last lines by n^{p-1} and summing implies the result, with $K = C_p\, E(\tau_+^{\,p})$.

Lemma 2: If $E(\tau_+^{\,p}) < \infty$ then $E(\tau(y)^p) \le K(y+1)^p \quad \forall\, y > 0$.

Proof: First notice that $E(\tau_+^{\,p}) < \infty \Rightarrow E(\tau(y)^p) < \infty, \forall\, y > 0$. For if $P\{X < 0\} > 0$ one conditions on the random walk at time 1 to show $E(t(\varepsilon)^p) < \infty$, $\exists\, \varepsilon > 0$, from which $E(\tau(y)^p) < \infty\ \forall y$ follows as below, if $P\{X < 0\} = 0$ the one-sided hitting problem is the same as a two-sided problem, for which Stein's Lemma (cf Feller [5], Sec. 18.2) says $\tau(y)$ has moments of all orders. To proceed with the proof, observe that for $K > 0$ an integer

$$\tau(Ky) \le \tau(y) + \tau(y)(\omega^+_{\tau(y)}) + \dots + \tau(y)(\omega^+_{\tau((K-1)y)})$$ and the $\tau(y)(\omega^+_{\tau(jy)})$ are i.i.d. Hence by Minkowski's Inequality.

$$E(\tau(Ky)^p) \le K^p\, E(\tau(y)^p)$$

and so

$$E(\tau(y)^p) \leq E\,\tau([y]+1)^p$$
$$\leq E(\tau(1)^p)([y]+1)^p$$
$$\leq E(\tau(1)^p)(y+1)^p$$

Lemma 3: For $x > 0$ let

$$R_{-x} = \begin{cases} -S_{t(-x)} - x\ , & t(-x) < \infty \\ 0\ , & t(-x) = \infty. \end{cases}$$

Then $\forall\ p \geq 1\ E((X^-)^p) < \infty \Rightarrow E(R_{-x}^{\,p-1};\ t(-x) < \infty) < K$, where K is independent of x.

Proof: This is essentially the same as Theorem 2.4 in Woodroofe [8].

$$P\{R_{-x} > y\} = \sum_{n=1}^{\infty} P\{t(-x) \geq n,\ S_n < -x - y\}.$$
$$\leq \sum_{n=1}^{\infty} P\{S_{n-1} \geq -x,\ S_n < -x - y\}$$
$$= \sum_{n=1}^{\infty} \int_{-x}^{\infty} F(-x-y-s)\ F^{*(n-1)}(ds)$$
$$= \int_{-x}^{\infty} F(-x-y-s)U\ (ds)$$
$$\leq C \sum_{k>-x} F(-x-y-k)$$
$$\leq C \sum_{k>0} F(-y-k)$$
$$\leq C \int_{-\infty}^{-y} F(z)dz.$$

where F^{*j} is the j-fold convolution of F with itself, and U is the renewal measure: $U(x) - U(y) = \sum_{m=0} [F^{*m}(x) - F^{*m}(y)]$. Multiplying the first and last statements by y^{p-2} and integrating gives the stated result.

<u>Lemma 4:</u> $E(|S_{\tau-}|^p; \tau- < \infty) < \infty \Rightarrow E((X^-)^{p+1}) < \infty \;\forall\; p > 0$.

<u>Proof:</u> The statement is invariant under change of scale, so if X is lattice one may assume that the span of X is less than 1. It may also be assumed that X is not bounded below, for otherwise the statement of the lemma is trivial. In this case, with

$$R_x = S_{\tau(x)} - x ,$$

since the asymptotic distribution of R_x has positive mass on $[0,1)$ (see Woodroofe [8], Sec. 2.2),

$$0 < r = \inf_{x \geq 0} P\{R_x < 1\}.$$

By time-reversal, for $n \geq 0$

$$\begin{aligned}
&P\{S_{\tau-} \in (-n-1, -n), \tau- < \infty\} \\
&= \sum_{j=1}^{\infty} P\{S_1 > 0, \ldots, S_{j-1} > 0, S_j \in (-n-1, -n)\} \\
&= \sum_{j=1}^{\infty} P\{S_j > S_1, \ldots, S_j > S_{j-1}, S_j \in (-n-1, -n)\}. \\
&\geq \sum_{j=1}^{\infty} P\{-n-1 > S_1, \ldots, -n-1 > S_{j-1}, S_j \in (-n-1, -n)\} \\
&= \sum_{j=1}^{\infty} P\{\tau_{-n-1} = j, R_{-n-1} < 1\} \\
&= P\{R_{-n-1} < 1\}.
\end{aligned}$$

And

$$P\{R_{-n-1} < 1\} \geq P\{R_{-n-1} < 1,\ X_1 \leq -n-1\}$$

$$= \int_{n+1}^{\infty} P\{R_{x-n-1} < 1\}\ P\{X_1^- \in dx\}.$$

$$\geq r\ P\{X_1^- > n+1\}.$$

Thus

$$P\{S_{\tau-} \in (-n-1, -n)\} \geq r\ P\{X_1^- \geq n+1\}.$$

Multiplying by n^p and summing gives the statement of the lemma.

Proof of the Theorem.

a => b $\quad E(\tau_+^p) < \infty \Rightarrow \quad E(\tau_-^{p-1};\ \tau_- < \infty) < \infty$. By a standard time reversal argument (See Feller [4], Sec. 12.2),

$$P\{\tau_+ > n\} = \sum_{j=1}^{\infty} P\{\tau_-^{(j)} = n\}.$$

In particular

$$P\{\tau_+ > n\} \geq P\{\tau_- = n\}.$$

Multiplying by n^{p-1} and summing gives the result.

b => a $\quad E(\tau_-^{p-1};\ \tau_- < \infty) < \infty \Rightarrow \quad E(\tau_+^p) < \infty$. Note that conditioned on $\tau_-^{(j)} < \infty \quad \tau_-^{(j)} = \sum_{i=1}^{j} Y_i$, where the Y_i are i.i.d. with

$P\{Y_i < y\} = P\{\tau_- < y \mid \tau_- < \infty\}$. Thus $\sum_{m=1}^{\infty} n^{p-1} P\{\tau_-^{(j)} = n\} =$
$E(|\tau_-^{(j)}|^{p-1};\ \tau_-^{(j)} < \infty) \leq j^{p-1}\ E(\tau_-^{p-1} \mid \tau_- < \infty)\ P\{\tau_- < \infty\}^j$

So

$$\sum_{m=1}^{\infty} n^{p-1} P\{\tau_+ > n\} \leq \sum_{j=1}^{\infty} j^{p-1} P\{\tau_- < \infty\}^{j-1} E(\tau_-^{p-1};\ \tau_- < \infty)$$

$$\leq K\ E(\tau_-^{p-1};\ \tau_- < \infty).$$

<u>a => c</u> $E(\tau_+^p) < \infty \Rightarrow E((X^-)^p) < \infty$.

By the Elementary Renewal Theorem (Chung [9], Thm. 5.5.2), $\exists\, c,\ K > 0$ such that $E(\tau(x)) > cx\ \forall\ x > K$. So $E(\tau(x)^p) \geq (E(\tau(x)))^p \geq c^p x^p\ \forall\ x > K$ Conditioning on the first step of the random walk gives

$$\begin{aligned} \infty > {} & E(\tau_+^p) \\ & \geq \int_0^\infty E(\tau(x))^p\ P\{X^- \in dx\}. \\ & \geq \int_K^\infty c^p\ x^p\ P\{X^- \in dx\}. \end{aligned}$$

The last line implies $E((X^-)^p) < \infty$.

<u>c => a</u> $E((X^-)^p) < \infty \Rightarrow E(\tau_+^p) < \infty$. It suffices to assume $X_i \leq c_x$ for some $c > 0$; for, X_i can be truncated above to give $\tilde{X}_i$ with $E\ \tilde{X}_i > 0$, and τ_+ for the random walk generated by the $\tilde{X}_i$ is larger than that for the X_i random walk, so if the claim can be proven for the $\tilde{X}_i$ process it follows for the X_i process.

In this case it may be assumed that the X_i have at least 2 moments. Wald's identity for the 2^{nd} moment gives

$$E(S_{\tau_+} - \mu\, \tau_+)^2 = (\mathrm{Var}\, X_i)\, E(\tau_+) < \infty.$$

But $S_{\tau_+} < c$ so $E\, S_{\tau_+}^2 < \infty \Rightarrow E\, \tau_+^2 < \infty$. Let $\hat{q} = \sup\{p \geq 2:$ $E((X^-)^q) < \infty \Rightarrow E(\tau_+^{\,q}) < \infty \;\forall\; 2 \leq q \leq p\}$. Suppose $\hat{q} < \infty$. Let $\hat{q} \leq q < 2\hat{q}$. Then $E(\tau_+^{\,q/2}) < \infty$. Therefore, by Burkholder and Gundy [2], Theorem 5.3.

$$E\, |\, S_{\tau_+} - \mu\, \tau_+\, |^q < \infty$$

from which $E\, \tau_+^q < \infty$ follows as above. This is a contradiction.

<u>d => b</u> $E(L(0)^{p-1}) < \infty \Rightarrow E(\tau_-^{\,p-1};\, \tau_- < \infty) < \infty.$

<u>Proof:</u> $L(0) \geq \tau_- 1_{\{\tau_- < \infty\}}$.

<u>a and b => d</u> $E(\tau_+^p) < \infty$, and $E(\tau_-^{\,p-1};\, \tau_- < \infty) < \infty \Rightarrow E(L(0)^{p-1}) < \infty.$

The idea of the proof is to express $L(0)$ as a sum of successive trips above and below the origin, until the random walk stays permanently above 0. Finding the random walk above 0 one must know the $p-1^{st}$ moment of the expected time to get back below 0 must be bounded no matter where the process is, provided it ever does. This is the content of Lemma 1. Having hit below 0 one must know that the $p-1^{st}$ moment of the expected time to reach 0 is finite. According to Lemma 2 this quantity is bounded by $K \int_0^\infty (|y| + 1)^{p-1}\, F(dy)$, where F denotes the hitting place of the non-positive axis.

Lemma 3 provides a uniform bound on the $p-1^{st}$ moments of these distributions F. The proof is then finished by observing that, because of the positive drift, this cycling behavior can only be repeated a few times.

<u>Proof</u>: Set $p = (1-q) = P\{\tau_- = \infty\}$. Define

$T_1 = \inf\{K \geq 1:\ S_K > 0$ and $\exists\ m < K$ with $S_m \leq 0$,

and for $n > 1$

$$T_n = \inf\{K > T_1 + \dots + T_{n-1} = S_K > 0 \text{ and } \exists\ T_1 + \dots + T_{n-1} < m < K \text{ with } S_m \leq 0\} - (T_1 + \dots + T_{n-1}),\ T_{n-1} < \infty$$

$$= \infty,\ T_{n-1} = \infty$$

[3.1] $$|L(0)|^{p-1} \leq \sum_{m=1}^{\infty} |T_1 + \dots + T_n|^{p-1}\, 1_{\{T_{n<\infty},\, T_{n+1=\infty}\}}.$$

$$E(|T_1 + \dots + T_n|^{p-1};\ T_{n<\infty},\ T_{n+1=\infty})$$
$$\leq E(|T_1 + \dots + T_n|^{p-1};\ T_n < \infty)$$
$$\leq n^{p-1}\, E(T_1^{p-1} + \dots + T_n^{p-1};\ T_n < \infty)$$

$E(T_i^{p-1};\ T_n < \infty)$ is estimated seperately when $i=n$, and $i<n$. First the case $i<n$.

$$E(T_i^{p-1},\ T_n < \infty) = E(E(T_i^{p-1};\ T_n < \infty | \mathfrak{F}_{T_1 + \dots + T_{n-1}}))$$
$$= E(T_i^{p-1};\ T_{n-1} < \infty\ P\{T_n < \infty | \mathfrak{F}_{T_1 + \dots + T_{n-1}}\};\ T_{n-1} < \infty)$$
$$\leq q\ E(T_i^{p-1};\ T_{n-1} < \infty) \qquad (*)$$

and

$$E(T_n^{p-1};\ T_n < \infty) = E(E(T_n^{p-1};\ T_n < \infty \mid \mathfrak{F}_{T_1 + \dots + T_{n-1}}))$$

$$= E(E^{S_{T_1 + \dots + T_{n-1}}}(T_1^{p-1};\ T_1 < \infty);\ T_{n-1} < \infty).$$

Consider for $x > 0$

$$E^x\{T_1^{p-1};\ T_1 < \infty\} = E^x\{(\tau_- + \tau(-S_{\tau-})(\omega^+_{\tau-}))^{p-1};\ \tau_- < \infty)$$

$$\leq 2^{p-1}(E^x(\tau_-^{p-1};\ \tau_- < \infty) + E^x(E^{S_{\tau-}}(\tau(0)^{p-1};\ \tau_- < \infty)).$$

The first term is $\leq$ K by Lemma 1. For the 2[nd], using lemmas 2 and 3 it follows that

$$E^x(E^{S_{\tau-}}(\tau(0)^{p-1};\ \tau_- < \infty))$$

$$= \int_{-\infty}^{0} E^y(\tau(0)^{p-1})\ P^x\{S_{\tau-} \in dy,\ \tau_- < \infty\}$$

$$\leq K' \int_{-\infty}^{0} |y + 1|^{p-1}\ P^x\{S_{\tau-} \in dy,\ \tau_- < \infty\}.$$

$$\leq K''.$$

Thus

$$E(T_n^{p-1};\ T_n < \infty) \leq K\ P\{T_{n-1} < \infty\}$$

$$\leq K\ q^{n-1}.\ (**).$$

Set $a_n = E(T_1^{p-1} + \dots + T_n^{p-1};\ T_n < \infty)$.

Summing (*) from 1 to n-1 and adding (**) gives

$$a_n \leq q\ a_{n-1} + K\ q^{n-1},$$

Therefore, a_n is geometrically decreasing, and $\Sigma\, a_n < \infty$. A look at 3.1 shows that $\Sigma\, a_n < \infty \Rightarrow E\, L(0)^{p-1} < \infty$.

<u>d => e</u> $E(L(0)^{p-1}) < \infty \Rightarrow E(N(0)^{p-1}) < \infty$

<u>Proof</u>: $L(0) \geq N(0)$.

<u>e => g</u> $E(N(0)^{p-1}) < \infty \Rightarrow E(|S_{\tau_-}|^{p-1};\ \tau_- < \infty) < \infty$

<u>Proof</u>: The amount of time spent getting back above ϕ after having hit below ϕ for the first time is $\tau(0)(\omega^+_{\tau_-})\, 1_{\{\tau_- < \infty\}}$. So

$$(1 + N(0))^{p-1} \geq \tau(0)(\omega^+_{\tau_-})\, 1_{\{\tau_- < \infty\}}.$$

and $\infty > E((1 + N(0))^{p-1}) \geq E(\tau^{p-1}(0)(\omega^+_{\tau_-});\ \tau_- < \infty)$

$$= E(E(\tau^{p-1}(0)(\omega^+_{\tau_-});\ \tau_- < \infty |\ F_{\tau_-})$$

$$= E(E^{S_{\tau_-}}(\tau(0)^{p-1};\ \tau_- < \infty) \Rightarrow E(|S_{\tau_-}|^{p-1};\ \tau_- < \infty) < \infty$$

as in the last part of a and b => d.

<u>g => f</u> $E(|S_{\tau_-}|^{p-1};\ \tau_- < \infty) < \infty \Rightarrow E(|S_{min}|^{p-1} < \infty)$

<u>Proof</u>: S_{min} can be written as Z_n, where Z_i is a random walk with $P\{Z_1 < y\} = P\{S_{\tau_-} < y\} | \tau_- < \infty\}$, $P\{M = n\} = P\{\tau_- < \infty\}^n\, P\{\tau_- = \infty\}$, n=0, 1, ..., and M is independent of the Z_i. This can be seen intuitively by considering the decreasing ladder process, or a quick proof can be based on a comparison of the characteristic functions given in Feller [4], Chapt. 18. $E(|S_{\tau_-}|^{p-1};\ \tau_- < \infty) < \infty \Rightarrow E(|Z_1|^{p-1}) < \infty$, so

$$\begin{aligned} E(|S_{min}|^{p-1}) &= \sum_n E(|Z_n|^{p-1})\ P\{M = n\} \\ &\leq E(|Z_1|^{p-1}) \sum_n n^{p-1}\ P\{M = n\} \\ &< \infty . \end{aligned}$$

$\underline{f \Rightarrow g}$ $E(|S_{min}|^{p-1}) < \infty \Rightarrow E\{|S_{\tau_-}|^{p-1};\ \tau_- < \infty\} < \infty$

Proof: $|S_{\tau_-}| 1_{\{\tau_- < \infty\}} \leq |S_{min}|$

$\underline{f \Rightarrow e}$ $E(|S_{min}|^{p-1}) < \infty \Rightarrow E(N(0)^{p-1}) < \infty.$

Proof: Since $E(|S_{min}|^{p-1}) < \infty$, then $E\{|S_{\tau_-}|^{p-1};\ \tau_- < \infty\} < \infty$. From lemma 4 $E((X^-)^{p+1}) < \infty$ so the result follows from $c \Rightarrow d \Rightarrow e$.

$\underline{f \Rightarrow c}$ $E(|S_{min}|^{p-1}) < \infty \Rightarrow E((X^-)^p) < \infty$

Proof: Follows from $f \Rightarrow g$ and Lemma 4.

4. Remarks and Applications.

Let Y_i be a i.i.d. sequence of random variables with $EY_i = 0$. Let $S_n = Y_1 + \ldots + Y_n$

$$L(\varepsilon) = \sum_{n=1}^{\infty} 1_{\{\sup_{j \geq n} \frac{|S_j|}{j} > \varepsilon\}},$$

$$N(\varepsilon) = \sum_{n=1}^{\infty} 1_{\{|\frac{S_n}{n}| > \varepsilon\}}.$$

Theorem 2: For $p \geq 2$

(1) $E((N(\varepsilon)^{p-1}) < \infty \Leftrightarrow E(|Y|^p) < \infty$

(2) $E(L(\varepsilon)^{p-1}) < \infty \Leftrightarrow E(|Y|^p) < \infty$

(3) $\sum_{n=1}^{\infty} P\{\sup_{j \geq n} |\frac{S_j}{j}| > \varepsilon\} \cdot n^{p-2} < \infty \Leftrightarrow E|Y|^p < \infty$

(4) $\sum_{n=1}^{\infty} P\{|\frac{S_n}{n}| > \varepsilon\} \Leftrightarrow E|Y|^2 < \infty.$

Remarks: The "only if" part of (4), for $p=1$ is due to Robbins and Hsu [6], (4) with $p=1$ is due to Erdos [3], (3) was first proved by Baum and Katz and can be found in [1].

Proof: (1) and (2) follow from the equivalence of c,d, and e by considering the random walks $S_n \pm n\varepsilon$. (3) is the same as (2) plus the observation that $P\{L(\varepsilon) > n\} = P\{\sup_{j \geq n} \frac{|S_j|}{j} > \varepsilon\}$, and (4) follows similarly from (1).

Let X_i be i.i.d. random variables with $E X_i = \mu \in (0,\infty)$, $S_n = X_1 + \ldots + X_n$.

Theorem 3: For $p \geq 1$ the following are equivalent:

(1) $E((X^-)^{p+1}) < \infty$

(2) $E(|S_{\tau_-}|^p; \tau_- < \infty) < \infty$

(3) $E(|S_{min}|^p) < \infty$.

Remark: The equivalence of (2) and (3) for $p=1$ is credited by Taylor [7], to unpublished work of Darling, Erdos and Kakutani, and Taylor adds a proof of the equivalence of (1). Kiefer and Wolfowitz [5] also credit the equivalence of (1) and (3) to unpublished results of Darling, Erdos and Kalutani; and they give their own proof. The moments of the minimum are of interest because the minimum has the distribution of the stationary distribution of a type of queueing proeess. See [4] p. 198.

Proof: This is the equivalence above, however, the tortuous path via the implications of Theorem 1 can be replaced by lemma 4.

I would like to thank Professor Siegmund for help received on this problem. In particular he showed me the time-reversal proof of b => a.

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Let S_n be a random walk with positive drift. Let $S_{min} = \inf_{n>0} \{S_n\}$. New proofs are given of the following: For $p \geq 1$ $E|S_{min}|^p < \infty \iff E(|S_1|^{p+1} 1_{(S_1<0)}) < \infty$; $\Sigma P\{S_n < 0\} < \infty \iff E(|S_1|^2 1_{(S_1<0)}) < \infty$, and some related results.

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